

# The mechanism of phase transitions of infinite-range systems enlightened by an elementary $\mathbb{Z}_2$ -symmetric classical spin model

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In this paper we present the most elementary model that we know with a first order  $\mathbb{Z}_2$ -symmetry breaking phase transition. It is a classical spin model with potential energy assuming only two values, which despite its dramatic simplicity reproduces all the characteristic features of a first order symmetry breaking phase transition. Its aim is not to reproduce anyway some physical systems, but to enlighten the mechanics at the deep origin of a phase transition of any system. We consider the model as elementary in this sense. Indeed, it reveals in the most evident way how the Boltzmann factor competes with the entropic factor in order to generate the phase transition by varying the temperature. The paper ends revisiting the solutions of the Ising model and the spherical model (Berlin-Kac) in the mean-field version which show the same picture of the model introduced in this paper, but extended to continuous phase transitions. A limit of this analysis is that all the three models here considered satisfies the strong constraint of having the potential expressible as a function of the magnetization, thus they belong to the class of infinite-range systems. Anyway, we hope that this study may be helpful in finding out the general sufficiency conditions under which a potential can entail a phase transition in a general physical system also in the finite-range case.

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## I. INTRODUCTION

Phase transitions are sudden changes of the macroscopic behavior of a physical system composed by many interacting parts occurring while an external parameter is smoothly varied, generally the temperature, but e.g. in a quantum phase transition is the external magnetic field. The successful description of phase transitions starting from the properties of the microscopic interactions between the components of the system is one of the major achievements of equilibrium statistical mechanics.

From a statistical-mechanical point of view, in the canonical ensemble, a phase transition occurs at special values of the temperature  $T$  called transition points, where thermodynamic quantities such as pressure, magnetization, or heat capacity, are non-analytic functions of  $T$ . These points are the boundaries between different phases of the system. Starting from the exact solution of the two-dimensional Ising model [17] by Onsager [27], these singularities have been found in many other models, and later developments like the renormalization group theory [15] have considerably deepened our knowledge of the properties of the transition points.

But in spite to the success of equilibrium statistical mechanics the issue of the deep origin of a phase transition remains open, and this motivates further studies of phase transitions. In this paper we present a  $\mathbb{Z}_2$ -symmetric classical spin model showing a first order symmetry breaking phase transition which, in our knowledge, is one of the most elementary models capable to show such a phenomenon in the canonical treatment. We think that in

it the general mechanism of entailing symmetry breaking phase transitions acts, and that it becomes manifest due to the extreme simplicity of the model. The model is derived from the "hypercubic model" introduced in [2] by replacing the real variables with classical spin ones, and it can be consider as a drastic simplification of the mean-field Ising model.

In more physically realistic models we think that the mechanism entailing phase transitions may be the same, but that it remains hidden due to the complexity of the models, also in that cases in which the canonical thermodynamic is analytically solvable. Indeed, even if the analytic solution is at disposal, we can find out whether a system undergoes to a phase transition or not only by analyzing the solution, but we do not know any general criterion capable to predict such a phenomenon founded only on the properties of the potential energy landscape.

Anyway we have to remark a limitation of the model we present, because being the potential writable as a function of the order parameter, a.i. the magnetization, the model has to be considered belonging to the class of infinite-range systems, which are not very realistic from the physical viewpoint, mostly in classical physics. Anyway, the mechanism entailing the phase transition may contain some features which may be in future generalized to the finite-range case. This is under study.

Returning to the potential, it is a double-well function of the magnetization  $m$ . Since the entropy  $s(m)$  is a concave function due to the geometry of configuration space, the phase transition emerges by the competition between the potential and the entropy in shaping the free energy  $f(m, T) = v - Ts$ , which can have one or two absolute minima with respect to the magnetization depending on the value of the temperature  $T$ . If  $T$  is less than the critical temperature,  $f$  becomes a double-well function

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as well the potential and the symmetry is broken, while if  $T$  is greater than the critical temperature,  $f$  remains a convex function as well the entropic term  $-Ts$  and the symmetry remains unbroken.

This picture of the free energy is exactly the one imagined by Landau-Ginzburg [15, 16] in order to describe phenomenologically the classical continuous phase transitions, but it can apply also to the first order ones as for the model of this paper.

Section II is devoted to the introduction of the new model and to its study in great details. Further, Section III, and IV are devoted to the Ising [17] and Berlin-Kac [3] models in the mean-field version revisited in order to show the occurrence of their well known classic continuous phase transitions by the same mechanism of the model introduced here.

## II. THE MODEL

In a recent paper [2] a simple model with first order  $\mathbb{Z}_2$ -symmetry breaking phase transition has been introduced. It has been called "hypercubic model" because the interaction among the  $N$  continuous degrees of freedom is by a double-well potential with a gap proportional to  $N$  constructed by hypercubes of  $\mathbb{R}^N$ .

From the hypercubic model we derive a more essential model by replacing the  $i$ th coordinate  $q_i$  with the classical spin  $\sigma_i \in \{-1, +1\}$ , the same of the Ising model. In order to reproduce the double-well potential we define the potential in the following way

$$V(\vec{\sigma}) = \begin{cases} -NJ & \text{if } \sigma_i = 1 \text{ and } \sigma_i = -1 \forall i \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

In this way the potential has a gap proportional to  $N$  which separates the two wells. The constant  $J$  plays the role of the strength of the interaction, and it modulates the critical temperature of the phase transition.

From a physical viewpoint such a potential may be regarded as the tendency of the spins to be in the two states  $\{1, \dots, 1\}$ ,  $\{-1, \dots, -1\}$  by a completely delocalized interaction among the spins proportional to their number  $N$ . This is similar to what happens in some quantum systems, where the non-locality of the wave function can produce particular kinds of interaction depending only on the number of particles and not on their distances.

The thermodynamic can be solved in one shot, by using the decomposition formula

$$Z_N = \sum_{\{\sigma\}} e^{-\frac{V(\vec{\sigma})}{T}} = \sum_{V_i} e^{-\frac{V_i}{T}} \omega_N(V_i) \quad (2)$$

where  $\omega_N(V)$  is the microcanonical density of states at fixed  $N$ , so

$$\begin{aligned} Z_N &= e^{-\frac{0}{T}} \omega(0) + e^{-\frac{NJ}{T}} \omega(-NJ) = \\ &= 2^N - 2 + 2e^{\frac{NJ}{T}} = e^{N \ln 2} - 2 + 2e^{\frac{NJ}{T}}. \end{aligned} \quad (3)$$

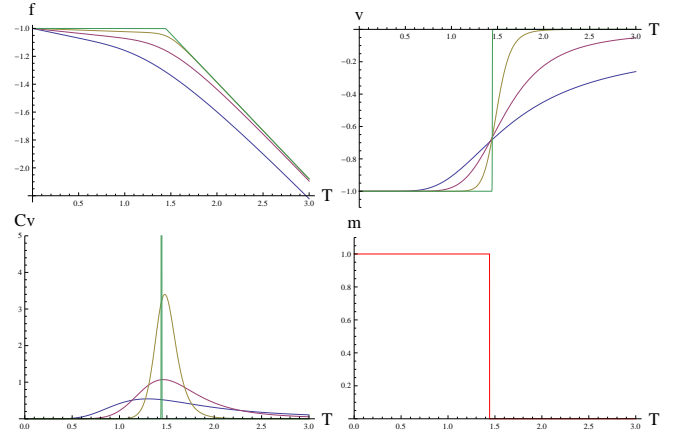


FIG. 1: From topo to bottom, and from left to right. Free energy, average potential, specific heat, and magnetization of the model introduced in section II for  $J = 1$ . The smooth lines are for  $N = 5, 10, 30$ , and show the nonuniform convergence toward a discontinuous limit corresponding to a first order phase transition with  $T_c = \frac{1}{\ln 2}$ .

In the limit of large  $N$  only one between the two exponentials of the right hand side of the last equation survives, so  $Z_N$  can be approximated as

$$Z_N \simeq \begin{cases} e^{N \ln 2} & \text{if } T \geq T_c \\ e^{\frac{NJ}{T}} & \text{if } T \leq T_c \end{cases} \quad (4)$$

where  $T_c = \frac{J}{\ln 2}$  is the critical temperature.  $T_c$  separates two different analytic forms of  $Z_N(T)$ .

In the thermodynamic limit the free energy, the average potential, and the specific heat are respectively

$$f = -\frac{T}{N} \ln Z_N = \begin{cases} -J & \text{if } T \leq T_c \\ -T \ln 2 & \text{if } T \geq T_c \end{cases} \quad (5)$$

$$\bar{v} = -T^2 \frac{\partial}{\partial T} \left( \frac{f}{T} \right) = \begin{cases} 0 & \text{if } T > T_c \\ -J & \text{if } T < T_c \end{cases} \quad (6)$$

$$c_v = \frac{\partial \bar{v}}{\partial T} = \begin{cases} 0 & \text{if } T > T_c \\ +\infty & \text{if } T = T_c \\ 0 & \text{if } T < T_c. \end{cases} \quad (7)$$

We can also give the relative expressions for finite  $N$

$$f_N = -\frac{T}{N} \ln Z_N = -\frac{T}{N} \ln \left( 2^N - 2 + 2e^{\frac{NJ}{T}} \right) \quad (8)$$

$$\bar{v}_N = -T^2 \frac{\partial}{\partial T} \left( \frac{f_N}{T} \right) = \frac{-2Je^{\frac{NJ}{T}}}{2^N - 2 + 2e^{\frac{NJ}{T}}} \quad (9)$$

$$c_{v,N} = \frac{\partial \bar{v}_N}{\partial T} = \frac{2J^2 N e^{\frac{NJ}{T}} (2^N - 2)}{(2^N - 2 + 2e^{\frac{NJ}{T}})^2 T^2}. \quad (10)$$

Now consider the magnetization. As  $T < T_c$  the average potential is  $\bar{v} = -J$ , so that the only configurations accessible to the representative point are  $\{1, \dots, 1\}$ , and  $\{-1, \dots, -1\}$ , at which the values of the magnetization per degree of freedom  $m = 1$ , and  $m = -1$  correspond, respectively. Since in the thermodynamic limit the probability of overturning simultaneously all the spins is vanishing, the magnetization has to be fixed in one of the two possible values.

As  $T > T_c$  the average potential is  $\bar{v} = 0$ , so that the representative point can freely go around the whole lattice expect the sites  $\{1, \dots, 1\}$ , and  $\{-1, \dots, -1\}$ . The magnetization is vanishing because in the thermodynamic limit the most probable configuration is the one with half spins 1, and half spins  $-1$ . Thus

$$m = \begin{cases} \pm 1 & \text{if } T < T_c \\ 0 & \text{if } T > T_c. \end{cases} \quad (11)$$

Summarizing, we are in front of the complete picture of a first order  $\mathbb{Z}_2$ -symmetry breaking phase transition, despite the dramatic simplicity of the model.

It is worth noticing that the potential (1) has to be considered as the effect of a completely delocalized interaction among the spins, thus it is infinite-range.

### A. Mapping in the class of the hypercubic models

We recall that the configuration space of the hypercubic model presented in [2] is made by a hypercube  $B$  centered in the origin of coordinates where the potential takes the value  $v_c > 0$ , except in two smaller hypercubes  $A^+$ , and  $A^-$  included in  $B$  where the potential takes the value 0. The hypercubes  $A^+$ , and  $A^-$  are such to be the images one of the other under  $\mathbb{Z}_2$ -symmetry. The model shows a first order symmetry breaking with  $T_c = v_c / \ln(b/a)$ , where  $b$  is the side of the hypercube  $B$ , and  $a$  is the side of the hypercubes  $A^+$ , and  $A^-$ . So, the class of the hypercubic models depends on two free parameters:  $v_c \in [0, +\infty)$ , and the quotient  $b/a \in [2, +\infty)$ . If  $b$  is strictly greater than  $2a$ , some freedom to dispose  $A^+$ , and  $A^-$  into  $B$  advances, but it does not affect the thermodynamic functions, but only the value of the magnetization.

The class of the models introduced in this section depends on the parameter  $J \in [0, +\infty)$ . The mapping from this class and the hypercubic models one can be made by identifying the lattice sites  $\{1, \dots, 1\}$ , and  $\{-1, \dots, -1\}$  with the centers of mass of the hypercubes  $A^+$ , and  $A^-$ . Since the statistical weight of a single lattice site is 1 and the weight of the whole lattice is  $2^N$ , the same proportion

has to hold between the volume of the hypercubes  $A^+$ , or  $A^-$ , and the hypercube  $B$ , so that  $b/a = 2$  has to be assumed. Finally, in order to complete the mapping we make the identification  $v_c \mapsto J$ .

The critical temperature results  $T_c = v_c / \ln(b/a) \mapsto 1/\ln 2$ , as it has to be, and the thermodynamic functions are the same.

Since the model of this section depends on one parameter only, it is in some extents more essential in what concern the mechanism giving rise to the phase transition with respect to the hypercubic model. Further, due to the discreteness of configuration space, it is possible to evaluate the effect of an external magnetic field, as we will see in Section II D.

### B. The free energy as a function of $m$ and $T$

In Section II we have calculated the partition function by decomposing it in a sum on the values of the potential which are 0,  $-NJ$ , but we can also choose to decompose it in a sum on the values of the magnetization, which will begin an integral in the limit  $N \rightarrow \infty$ . In this way we can deduce the shape of the free energy with its minima as a function of the magnetization, and moreover we can evaluate the effect of an external magnetic field applied to the system.

Let  $k$  be a positive integer which counts the number of spins  $\sigma_i = -1$  in a configuration of the system, so  $0 \leq k \leq N$ .  $k$  labels the subsets of configuration space  $\{-1, 1\}^N$  at constant magnetization  $\Sigma_k$  defined as

$$\Sigma_k = \left\{ \bar{\sigma} \in \{-1, 1\}^N : \frac{1}{N} \sum_{i=1}^N \sigma_i = \frac{k}{N} \right\}. \quad (12)$$

The configuration space is reconstructed by the disjointed union of all the  $\Sigma_k$ 's:  $\{-1, 1\}^N = \bigcup_{k=0}^N \Sigma_k$ . The relation which links  $k$  to the corresponding magnetization  $m_k$  is  $m_k = 2k/N - 1$ , so that  $-1 \leq m_k \leq 1$ .

The potential can be written as a function of  $k$

$$V_k = \begin{cases} -NJ & \text{if } k = 0, N \\ 0 & \text{if } 0 < k < N \end{cases} \quad (13)$$

and so the partition function can be decomposed as a summation over all the values of  $k$

$$Z_N = \sum_{k=0}^N e^{-\beta V_k} \text{vol}(\Sigma_k) \quad (14)$$

where  $\text{vol}(\Sigma_k)$  ("vol" stand for "volume", even though this word generally refers to continuous sets, but with a small abuse of language we use it also for discrete sets. A more appropriate word may be "cardinality") is the number of lattice sites belonging to  $\Sigma_k$ .  $\text{vol}(\Sigma_k)$  is nothing

but the density of states at fixed magnetization, which is linked to the entropy density at fixed magnetization  $s_k$  by the relation  $\text{vol}(\Sigma_k) = e^{Ns_k}$ .

It turns out that

$$\text{vol}(\Sigma_k) = \binom{N}{k} = \frac{N!}{k!(N-k)!} \quad (15)$$

where  $\binom{N}{k}$  are the binomial coefficients. So

$$\begin{aligned} Z_N &= \sum_{k=0, N} e^{N\beta J} + \sum_{k=1}^{N-1} \frac{N!}{k!(N-k)!} = \\ &= 2e^{N\beta J} + \sum_{k=1}^{N-1} e^{\ln \frac{N!}{k!(N-k)!}}. \end{aligned} \quad (16)$$

In order to search for the thermodynamic limit, we can use the Stirling formula to approximate the factorials in the binomial coefficients for large  $N$

$$\frac{N!}{k!(N-k)!} \simeq \left( \frac{(2\pi)^3 N}{k(N-k)} \right)^{\frac{1}{2}} \left( \frac{1}{1 - \frac{k}{N}} \left( \frac{1}{\frac{k}{N}} - 1 \right)^{\frac{k}{N}} \right)^N \quad (17)$$

Recalling that  $\frac{k}{N} = \frac{m_k+1}{2}$ , we can express  $Z_N$  in terms of the magnetization  $m_k$ , and then make the substitution  $m_k \mapsto m$  where  $m$  is a continuous real variable belonging to the interval  $[-1, 1]$ . This substitution is possible because as  $N \rightarrow \infty$  the values of  $m_k$  becomes dense and equally spaced in the interval  $[-1, 1]$ . Thus

$$\begin{aligned} Z_N &\simeq \int_{-1}^{+1} dm e^{-\beta V(m)} \text{vol}(\Sigma_m) = \\ &= \sum_{m=-1, +1} e^{N\beta J} + \int_{-1}^{+1} dm e^{N \ln \left( \frac{2}{1-m} \left( \frac{1-m}{1+m} \right)^{\frac{1+m}{2}} \right)}. \end{aligned} \quad (18)$$

The potential, and the entropy densities as functions of  $m$  remain defined respectively as follows

$$v = \begin{cases} -J & \text{if } m = \pm 1 \\ 0 & \text{if } -1 < m < +1 \end{cases} \quad (19)$$

$$s = \ln \left( \frac{2}{1-m} \left( \frac{1-m}{1+m} \right)^{\frac{1+m}{2}} \right). \quad (20)$$

The free energy  $f = v - Ts$  is plotted in Figure 2.

The central minimum of  $f$  is  $-T \ln 2$  which competes with the two other extreme minima  $-J$  in order to determine the absolute minima, and so giving rise to the critical temperature of the system  $T_c = \frac{J}{\ln 2}$ . The magnetization is already given in (11).

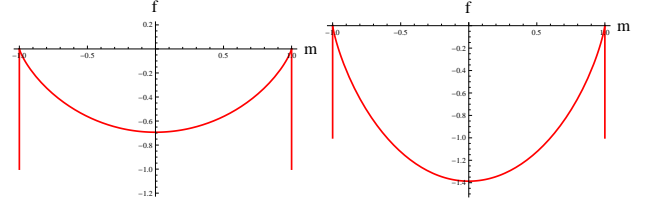


FIG. 2: Free energy as a function of  $m$  as  $T < T_c$  (left), and as  $T > T_c$  (right).

*Remark.* The fact that the potential (1) is a function of  $m$  lets to separate the contribute of the Boltzmann factor and the density of states at fixed  $m$  in the calculation of the partition function. Since the density of states increases as  $e^N$  due to the dimension of configuration space which increases as  $N$ , setting the jump proportional to  $N$  is a necessary and sufficient condition in order to entail the phase transition. Indeed, if the gap increased faster than  $N$  the potential would not be intensive, while if the gap increased more slowly than  $N$  the phase transition would disappear.

### C. Fisher zeros

The fact that the partition function can be exactly solved allows us to discuss the exact localization of the Fisher zeros [12], the zeros of the partition function in the complex temperature plane. Because the analyticity of  $Z_N$ , the zeros have not to lie on the real axis, but if a singularity is present in  $Z$  in the limit  $N \rightarrow \infty$ , they have to converge to the real axis in correspondence of the critical temperature. The Fisher zeros are the analog of the zeros of the grand canonical partition function in the complex fugacity plane introduced by Yang and Lee in [25].

We introduce the inverse temperature  $\beta = 1/T$  and solve the equation

$$Z_N(\beta) = 0 \quad (21)$$

whose solutions are

$$\beta_0 = \pm \frac{i(2k+1)\pi}{NJ} + \frac{1}{NJ} \ln(2^{N-1} - 1), \quad k \in \mathbb{N}. \quad (22)$$

As we expected, the limit  $N \rightarrow \infty$  of each solution is the inverse critical temperature

$$\lim_{N \rightarrow \infty} (\beta_0) = \beta_c = \frac{\ln 2}{J}. \quad (23)$$

### D. Effect of an external magnetic field

In this section we determine the effect of an external magnetic field applied to the system. The new potential has to take into account the magnetic interaction by the term

$$-\sum_{i=1}^N \sigma_i H = -NmH \quad (24)$$

so the new potential is

$$V_H = Nv - NmH \quad (25)$$

where  $v$  is given in (19). The new free energy is therefore

$$f_H = v - Ts - mH \quad (26)$$

where  $s$  is given in (20).  $f_H$  is plotted in Figure 3.

In order to find out the magnetization we have to minimize  $f_H$  with respect to  $m$ . We start by solving the following equation

$$\frac{\partial f_H}{\partial m} = -\frac{T}{2} \ln \left( \frac{1-m}{1+m} \right) - H = 0 \quad (27)$$

whose solution is

$$m_{cent} = \tanh \left( \frac{H}{T} \right). \quad (28)$$

Then we confront  $f_H(m_{cent})$  with  $f_H(1)$  if  $H > 0$ , or with  $f_H(-1)$  if  $H < 0$ . This can be easily made by introducing the quantity

$$\begin{aligned} \Delta f_H &= f_H(m_{cent}) - f_H(\pm 1) \\ &= -T_c \ln \left( 1 + e^{\pm \frac{H}{T}} \right) \pm (2H + J) = 0. \end{aligned} \quad (29)$$

respectively if  $H \gtrless 0$ . The root gives a sort of pseudo-critical temperature  $T_c(J, H)$  which separates two different analytic forms of the magnetization

$$m = \begin{cases} \pm 1 & \text{if } T < T_c(J, H) \\ \pm \tanh \left( \frac{H}{T} \right) & \text{if } T > T_c(J, H). \end{cases} \quad (30)$$

In the limit  $H \mapsto \pm\infty$ ,  $T_c(J, H) \mapsto \infty$ , so the magnetization remains frozen in  $m = 1$  or  $m = -1$  for all values of  $T$ .

We observe that the presence of the external field breaks the symmetry of the system, but does not eliminate the phase transition meant as the presence of a nonanalyticity in the thermodynamic function. This is not surprising, and it is due to the fact that the phase transition is of the first order, and not continuous.

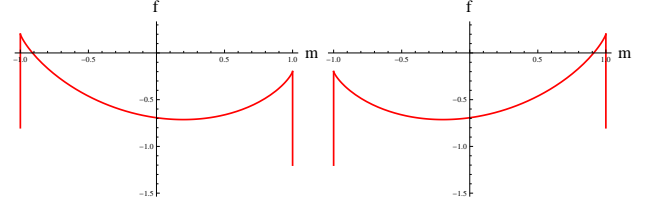


FIG. 3: Effect of an external field on the free energy (26) as  $H > 0$  (left), and as  $H < 0$  (right). The temperature is below  $T_c$ .

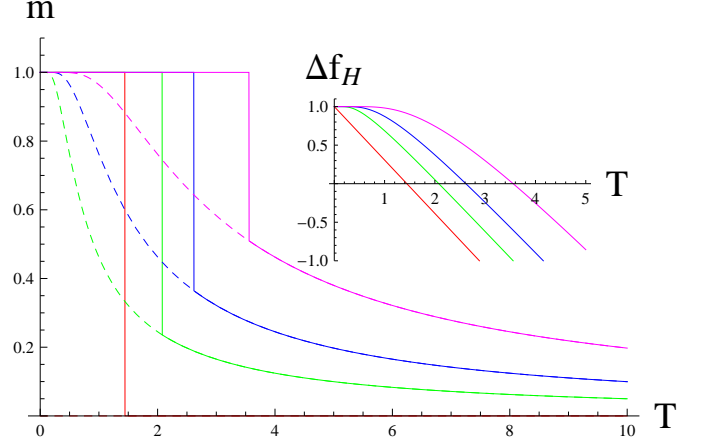


FIG. 4: Magnetization as a function of  $T$  for some values of the external field  $H = 0, 0.5, 1, 2$  (red, green, blue, magenta).  $J = 1$  (continuous lines), and  $J = 0$  (dashed lines). In the inset the function (29), whose root is the pseudo-critical temperature  $T_c(J, H)$  which separates the two analytic forms of the magnetization.

### III. THE MEAN-FIELD ISING MODEL

The solution of the classical mean-field Ising model is by means of the mean-field theory and it is a paradigmatic example in literature, e.g. [15, 16]. Nevertheless, we revisited it enlightening how it lets to separate the contribution of the potential from the entropy in the free energy at fixed magnetization, showing as they compete in order to entail the classical second order phase transition occurring in this model.

The potential is as follows

$$V = -\frac{J}{N} \sum_{i,j=1}^N \sigma_i \sigma_j \quad (31)$$

where  $\sigma_i \in \{-1, 1\}$ ,  $J > 0$ , and the factor  $1/N$  is introduced in order to guarantee the intensivity of the potential per degree of freedom.

Following the same method already used to solve the model in Section II, we introduce the positive integer  $k$  which labels the subsets of the configuration space

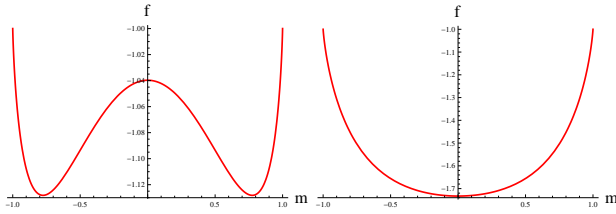


FIG. 5: Free energy as a function of the magnetization of the mean-field Ising model (35) as  $T < T_c$  (left), and as  $T > T_c$  (right).

$\{-1, 1\}^N$  at constant magnetization  $m_k$  whose elements have  $k$  positive spins.  $V$  can be easily written as a function of  $k$  because the interaction among the spins is "all with all"

$$V_k = -\frac{J}{N}(N - 2k)^2 = -JN \left(1 - \frac{2k}{N}\right)^2. \quad (32)$$

Since  $k$  is linked to the magnetization by the relation  $m_k = 2k/N - 1$ , the potential can be expressed as a function of  $m_k$

$$V_k = -JNm_k^2. \quad (33)$$

Then, as  $N \rightarrow \infty$  we can make the substitution  $m_k \rightarrow m$  where  $m \in [-1, 1]$  is a continuous real variable, getting finally

$$v = \frac{V}{N} = -Jm^2. \quad (34)$$

The entropy is the same of the model in Section II already found out in (20). So the free energy results to be

$$f = -Jm^2 - T \ln \left( \frac{2}{1-m} \left( \frac{1-m}{1+m} \right)^{\frac{1+m}{2}} \right). \quad (35)$$

$f$  is plotted in Figure 5.

In order to find out the spontaneous magnetization we have to differentiate  $f$  with respect to  $m$  and set to zero, the resulting equation gives  $m(T)$  in the implicit form

$$m = \tanh \frac{2Jm}{T}. \quad (36)$$

The last equation has two symmetric solutions as  $T < 2J$ , and 0 as  $T \geq 2J$ , so the critical temperature is  $T_c = 2J$ , as well known.

We do not add nothing about the effect of an external field because it is similar to what happens in the case of model in SEC. II, and because it is well known in literature, e.g. [16].

#### IV. THE MEAN-FIELD SPHERICAL MODEL

In this Section we add another example of a mean-field model solvable in the same way applied in previous Sections. It is the model introduced by Berlin and Kac [3, 19], and also known as the *spherical model*. It approximates the feature of the Ising model by substituting the discrete classical spin variables  $\sigma_i$  with the continuous real  $q_i$ , and constrained them on a  $N$ -sphere of radius  $\sqrt{N}$ , where  $N$  is the number of degrees of freedom. Such an  $N$ -sphere contains the lattice sites of the Ising model if  $\sigma_i \in \{-1, 1\}$ .

The potential is the same of the Ising model

$$V = -\frac{1}{N} \sum_{i,j=1}^N J_{ij} q_i q_j \quad (37)$$

where the sum is extended up all the couples of variables because we consider only the ferromagnetic mean-field version, and so we also set  $J_{ij} = J > 0$ . The factor  $1/N$  has been introduced in order to guarantee the intensivity of the potential per degree of freedom.

Thanks to this choice,  $V$  can be written as a function of the magnetization  $m = 1/N \sum_{i=1}^N q_i$

$$v = \frac{V}{N} = -\frac{J}{N^2} \left( \sum_{i=1}^N q_i \right)^2 = -Jm^2 \quad (38)$$

and so the free energy as a function of  $m$  and  $T$  can be derived. In order to do this, we want to calculate the measure of  $\Sigma_m$  which is the intersection between the hyperplane  $\sum_{i=1}^N q_i = Nm$  and the  $N$ -sphere of radius  $\sqrt{N}$ . Thus,  $\Sigma_m$  is an  $(N-1)$ -sphere and its volume is

$$\text{vol}(\Sigma_m) = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} r(m)^{N-2} \quad (39)$$

where the radius  $r(m)$  is a function of the magnetization.  $r$ , by the Pythagorean theorem, is

$$r(m) = \sqrt{N}(1 - m^2)^{\frac{1}{2}} \quad (40)$$

where  $m$  belongs to the interval  $[-1, 1]$ . The sketch in Figure 6 can help the comprehension.

The entropy results to be

$$s_N = \frac{1}{N} \ln \text{vol}(\Sigma_m) \quad (41)$$

and by applying the Stirling approximation of  $N!$  we obtain in the limit  $N \rightarrow \infty$

$$s(m) = \frac{1}{2} \ln(1 - m^2) + \frac{1}{2} \ln \sqrt{2\pi e}. \quad (42)$$

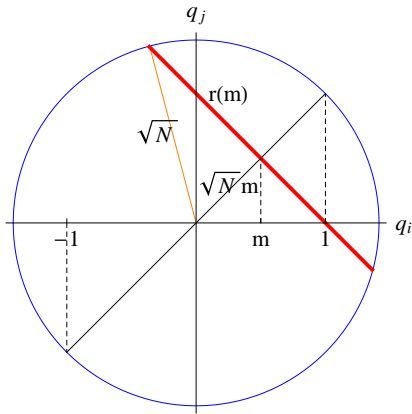


FIG. 6: Mean-field spherical model. The circle represents the configuration space schematically drawn at  $N = 2$ , and the secant (red) represents the  $\Sigma_m$  whose radius  $r$  is linked to the magnetization  $m$  via the Pythagorean theorem applied to the triangle sketched.

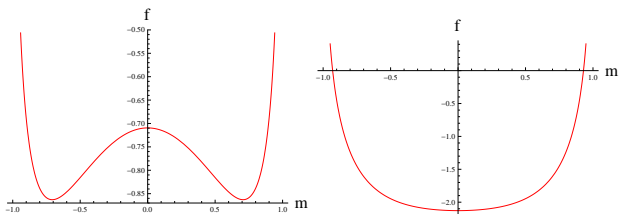


FIG. 7: Free energy as a function of the magnetization of the mean-field spherical model (43) as  $T < T_c$  (left), and as  $T > T_c$  (right).

Finally the free energy  $f = v - Ts$  results

$$f = -Jm^2 - \frac{T}{2} \ln(1 - m^2) - \frac{T}{2} \ln \sqrt{2\pi e}. \quad (43)$$

By minimizing with respect to  $m$  we find out  $m(T)$

$$m = \begin{cases} \pm (1 - \frac{T}{2J})^{\frac{1}{2}} & \text{if } T \leq T_c \\ 0 & \text{if } T \geq T_c \end{cases} \quad (44)$$

where  $T_c = 2J$  is the critical temperature.  $m(T)$  shows the well known continuous phase transition between the ferromagnetic phase and the paramagnetic one. By inserting it in (43) we obtain the free energy

$$f = \begin{cases} -J + \frac{T}{2} - \frac{T}{2} \ln \left( \frac{T}{2J} \right) & \text{if } T \leq T_c \\ 0 & \text{if } T \geq T_c \end{cases} \quad (45)$$

which shows a discontinuity in the second derivative at  $T = T_c$ , coherently with the continuity of the phase transition.

The picture of the phase transition is identical to the one of the mean-field Ising model apart non-substantial differences in the shape of the entropy.

## V. CONCLUDING REMARKS

In this article we have introduced a classical spin model showing a first order  $\mathbb{Z}_2$ -symmetry breaking phase transition. This model can be considered a drastic simplification of the mean-field Ising model [17] which conserves the minimal conditions sufficient to induce a phase transition. These conditions are a double-well potential as a function of the magnetization  $m$  with the gap between the absolute minima proportional to the number of degrees of freedom  $N$ . The difference between the model here introduced and the mean-field Ising model is the type of phase transition: first order in the former, continuous in the latter. This is due to the discontinuous and continuous behavior of  $v(m)$  in the former and in the latter model, respectively.

The double-wellness of the potential opposes to the concavity of the entropy of configuration space in determining the shape of the free energy  $f(m, T) = v(m) - Ts(m)$ . Indeed, the competition between these two shapes takes place giving rise to a double-well  $f(m)$  or a convex  $f(m)$  depending on the value of the temperature  $T$ : if  $T < T_c$  then the shape of the potential wins and so the symmetry is broken, conversely if  $T > T_c$  then the shape of the entropy wins and so the symmetry remains unbroken [15, 16].

This analysis has been possible due to the fact that the potential can be written as a function of  $m$ , more generally the order parameter. In this way the tendency to the symmetry breaking induced by the double-well potential and the tendency to leave unbroken the symmetry by the entropy of configuration space have been able to be separated.

We think that this mechanism has good probabilities to be the general one acting whenever a symmetry breaking phase transition occurs, but we have to take into account that the potential of the model studied in this paper are infinite-range. This is not the most general case occurring in general physical systems, especially in classical physics. Nevertheless, there may be the way to generalize the mechanism to finite-range systems. This is under study.

The paper ends with the revisiting the solution of the Berlin-Kac model [3, 19] (known also as the *spherical model*) in the mean-field version. The study reveals the same picture already found out in the mean-field Ising model, apart non-substantial differences in the shape of the entropy.

If the potential is not expressible as a function of  $m$ , but always infinite-range there are evidences that a double-well potential with a gap proportional to  $N$  can be again at the base of a  $\mathbb{Z}_2$ -symmetry breaking phase transition, as suggested by the studies contained in [1, 2]. In that papers some models with first order and continuous phase transition are built by modeling a double-well potential generalized to  $N$ -dim with gap proportional to  $N$ .

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